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Synchronization in small assemblies of chaotic systems

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(Received 10 October 1995)

In this paper we investigate the behavior of small networks of van der Pol–Duffing chaotic oscillators that are connected through local, although not weak, coupling through a recently introduced [Phys. Rev. E **52**, R2145 (1995)] extension of the Pecora-Carroll synchronization method. The method allows one to design a variety of settings with different ways of connecting a number of low-dimensional circuits. It is shown that a variety of different behaviors can be obtained, depending, among other factors, on the symmetry of the connections and on whether the oscillators are identical or different. One may cite the emergence of coherent behavior (a single cluster), either chaotic or periodic, as stemming just from interaction among the different chaotic units, although several coexisting clusters are found for other settings.

PACS number(s): 05.45.+b

I. INTRODUCTION

At first sight a chaotic system should be one defying any attempt of synchronization, due to its strong dependence on the initial conditions that would amplify any small disturbance. However, it has been suggested [1], and later proved [2], that this is not the case, and that chaotic systems may synchronize in some circumstances. One of the possibilities is one-way coupling, in which one has a drive-response couple, such that a signal coming from the drive makes the response synchronize without influencing the drive. This has been shown in a seminal contribution by Pecora and Carroll (PC) [2], including the application to electric circuits. The PC method is specially suited for the case of analog circuits, as one-way connections can be easily implemented through operational amplifiers. A problem is that drive and response must have a part in common, implying from this fact that the response has, in practice, a reduced dimensionality compared to the drive.

An interesting extension of these ideas is to the case of a cascade [3], in which the response of some drive system acts, in turn, as the drive of a second response system. At variance with what happens with a single connection, in one such cascade the input signal is regenerated by the second response circuit. By taking advantage of the degree of stochasticity that a chaotic signal exhibits to a casual observer, it has been suggested that these cascades could act as chaotic filters, potentially useful in the field of secure communications [4,5]. The idea is to mask the information to be transmitted with the chaotic signal, the main advantage being that one could *bury* the information to be transmitted in the chaotic signal by 30–40 dB, something that would not be possible with classical masking techniques based on the use of purely stochastic masks. See, however, Ref. [6] for recent evidence that the kind of masking to be used needs to be more sophisticated than the simple technique just sketched.

If a cascade is able to regenerate a given input signal, then a network formed by many low-dimensional chaotic systems could be useful as an information processing system. In this sense, this would be a generalization of the so-called cellular neural networks [7], devices able to perform useful computations based just on bistable circuits. Chaotic networks could operate by performing transitions among different kinds of coherent states. In addition, physiologists have found such coherent waves as evoked responses in the brains of some mammals [8], and have speculated [9] on their role in perception and other mental capabilities.

The aim of the present contribution is to suggest a way of setting up complex networks composed of many low-dimensional chaotic circuits. The PC method, in the form introduced in the original contribution [2], presents some limitations to the achievement of this goal, due to the fact that the drive and response systems share the driving signal. Although other versions of the method that overcome these limitations have already appeared (see, e.g., Ref. [10]), here we shall resort to an extension [11] of the PC method in which the input signal enters at a precise location of the dynamical equations of the response system. The result is that one is able to regenerate the input signal within a single connection, while the possibility of using the same connection twice without alternating with the other one is allowed. The result is that a given system may receive input from a number of systems, and that a number of different networks with different connectivities can be set up, thus resembling the situation found in biological neural networks.

All the results presented in the present contribution have

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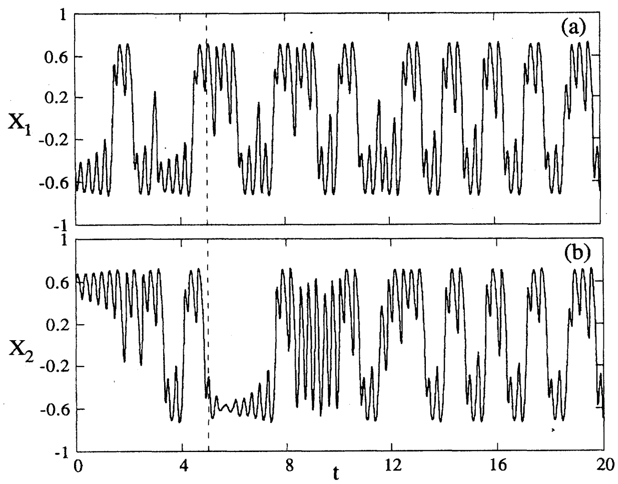

FIG. 1. Synchronized behavior obtained by connecting two identical van der Pol–Duffing circuits (1) in the chaotic regime using the connection defined by Eq. (2). The parameters for the circuit are $\nu=100$, $\alpha=0.35$, and $\beta=450$.

been obtained for the case of four van der Pol–Duffing systems [12] in the chaotic state. Although this is a fairly small number, it has been possible to observe the appearance of different situations like coherent chaotic and periodic states, and states in which smaller clusters occur. However, it is difficult to know whether the appearance of collective behavior will be a feature of larger networks of chaotic oscillators connected through this kind of local, although not weak, coupling, and what will happen in the thermodynamic limit. Kaneko [13] has performed a number of very interesting studies in the case of globally coupled maps, in which the local units are quadratic or circle maps in the chaotic state, that can be taken as mean-field-type representations of

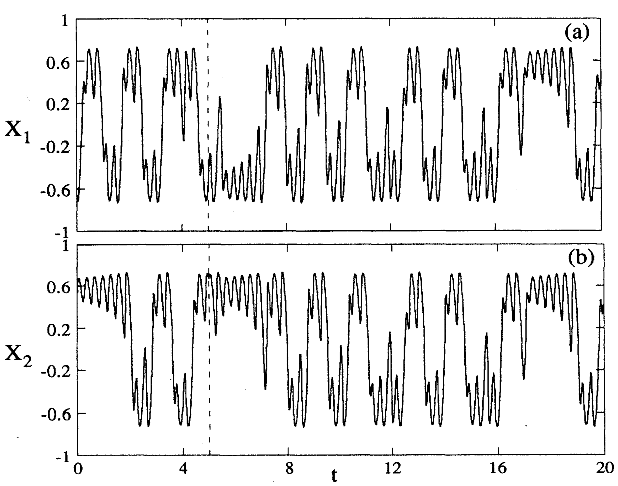

FIG. 2. Synchronized behavior obtained by connecting two identical van der Pol–Duffing circuits (1) in the chaotic regime through connection (3). See Fig. 1 for the parameters.

TABLE I. Lyapunov exponents for the van der Pol–Duffing system (1) (first line), and transverse Lyapunov exponents for various connections of two Van der Pol–Duffing systems (the corresponding equation numbers are given).

Connection	Lyapunov exponents
(1)	(1.29,0.00, −49.41)
$x(t)$ (2)	(−0.50, −0.50, −52.09)
$\overline{\nu y(t)}$ (3)	(−0.50, −0.50, −52.09)
$-\overline{\nu(x(t)^3-\alpha x(t))}$ (4)	(0.00, −0.50, −0.50)
$-z(\overline{t)}$ (5)	(4.47,0.00, −52.60)
$\nu\alpha\overline{x(t)}$ (6)	(0.24,0.22, −83.58)

coupled map lattices, which are useful models for spatiotemporal phenomena [14]. In these studies he has observed interesting phenomena, like the appearance of coherent, ordered, partially ordered, and turbulent phases, and also the violation of the law of large numbers [15] in the turbulent phase, this stemming from the presence of some kind of

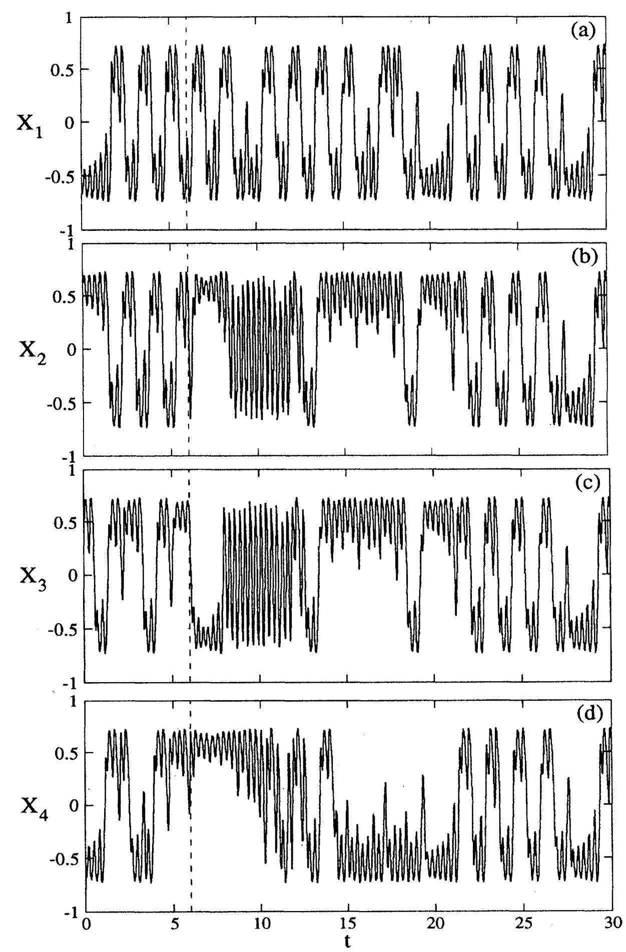

FIG. 3. Cascade of four van der Pol–Duffing systems (1) where every two contiguous systems are connected through (2), the network being defined by Eq. (7). See Fig. 1 for the parameters.

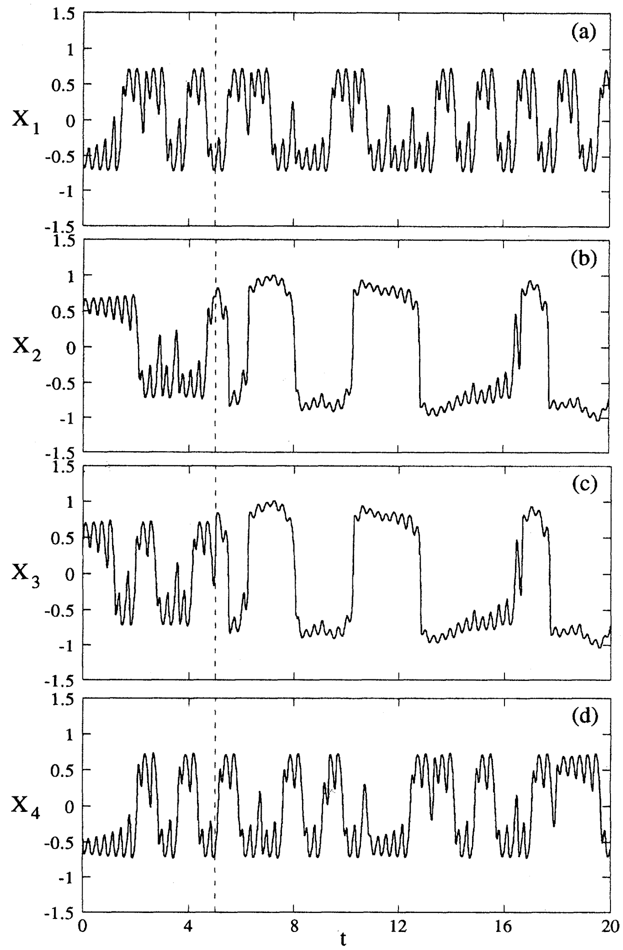

FIG. 4. Four van der Pol–Duffing systems (1) connected in competition according to Eq. (8), such that the first oscillator drives the second and third systems through connection (1), while in turn the fourth oscillator also drives the second and third ones through connection (5). See Fig. 1 for the parameters.

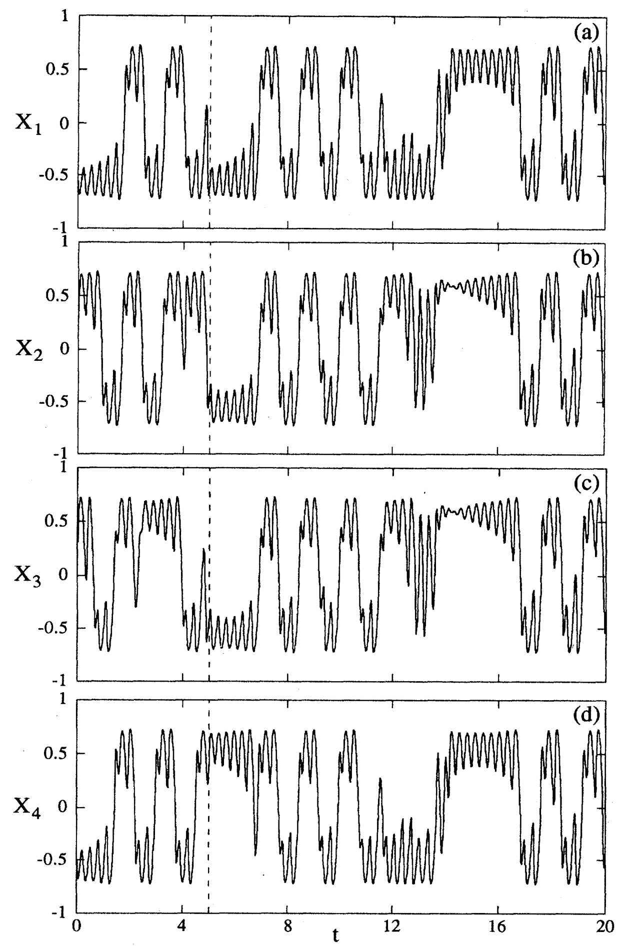

FIG. 5. Four van der Pol–Duffing oscillators (1) connected in a closed loop according to Eq. (9), such that connections (2) and (3) alternate. See Fig. 1 for the parameters.

unexpected order. The situation in which several phases may coexist seems particularly promising for the storage of information and modeling of some biological phenomena, like cell division and differentiation [16]. Another interesting topic that has aroused some interest in the last years is the appearance of collective behavior in high-dimensional discrete systems with synchronous updating, like cellular automata [17] with a totalistic rule of the game of life type [18] and also in coupled map lattices [19] with the same type of coupling. The explanation of this behavior is still under debate [20], and thus theoretical explanations are called for.

The structure of the present paper is as follows. In Sec. II the model chosen to represent the chaotic units is discussed, together with the method used to connect these systems. Then, Sec. III contains the main results of the present contribution, both for networks composed of identical units and for networks for which some units are described by different parameters. Finally, Sec. IV gathers the main conclusions of the present work.

II. MODEL AND METHOD

In the present work we shall consider the Van der Pol–Duffing (VPD) oscillator, already used for synchronization purposes in Ref. [12],

$$\dot{x}_1 = -\nu(x_1^3 - \alpha x_1 - y_1), \quad \dot{y}_1 = x_1 - y_1 - z_1, \quad \dot{z}_1 = \beta y_1 . \tag{1}$$

This circuit can be easily implemented in the form of an analog circuit [21] that bears a close resemblance to Chua's circuit [5], the main difference being that the piecewise linear element of the latter is replaced by a cubic nonlinearity that is quite easily implemented in practice by using a set of diodes and an operational amplifier.

Considering β as the bifurcation parameter the double-minimum shape of the Duffing system makes the system have two alternative minima for $\beta > \sim 1050$ that exhibit independently a period-doubling route to chaos when one goes to lower values of β. The two strange attractors born in this process are not connected, although they merge in an attractor-merging crisis [22] for smaller values of β. For even smaller values of β ($\beta \sim 650$) a periodic window is created in a saddle-node bifurcation. This periodic orbit extends to both minima and is stable down to $\beta \sim 500$, where another period-doubling cascade starts, ending abruptly in an interior crisis [22] when the stable and unstable orbits created in the saddle-node bifurcation collide. After this set of processes the system becomes once again chaotic, and, thus, we have chosen to work at $\beta = 450$. There is another feature of this system that is important in other behaviors to be found in other sections. It is the fact that the system exhibits a competing attractor, namely a stable limit cycle. Depending on the initial condition the system will be described by either attractor, while it can be assured that that initial conditions close to the origin belong to the attraction basin of the strange attractor, as the limit cycle surrounds the strange attractor.

The low-dimensional units have been connected using a recently introduced modification [11] of the Pecora-Carroll method [2]. The main idea of this modified method is to introduce the driving signal at a given place of the evolution equations of the response, the main difference with the PC method being that in the latter the driving signal is introduced at *all* possible places, suppressing completely the dynamical evolution of the variable that is homologous to the driving signal in the response dynamical system. The result is that the driving signal is regenerated by the response circuit.

In particular, two stable connections have been found in the case of the Van der Pol–Duffing system (1). In the first one of them, and considering (1) as the evolution equation for the drive, the response is written as

$$\dot{x}_2 = -\nu(x_2^3 - \alpha x_2 - y_2), \quad \dot{y}_2 = \underline{x_1(t)} - y_2 - z_2, \quad \dot{z}_2 = \beta y_2, \tag{2}$$

while an example in which synchronization behavior appears is shown in Fig. 1. Notice that the place where the driving signal enters has been emphasized by underlining it, while the dependence on t stresses its role as an external forcing parameter. The second stable connection from the viewpoint of synchronization can be written in the form

$$\dot{x}_2 = -\nu[x_2^3 - \alpha x_2 - \underline{y_1(t)}], \quad \dot{y}_2 = x_2 - y_2 - z_2, \quad \dot{z}_2 = \beta y_2, \tag{3}$$

while an example is shown in Fig. 2. On the other hand, the following connection has been found to be marginally stable:

$$\dot{x}_2 = -\nu[\underline{x_1(t)}^3 - \alpha \underline{x_1(t)} - y_2], \quad \dot{y}_2 = x_2 - y_2 - z_2, \dot{z}_2 = \beta y_2, \tag{4}$$

which can be easily checked out as the response becomes a forced *linear* system. The following connections are not synchronizing, but they are being reported because they lead to some interesting behavior to be analyzed later:

$$\dot{x}_2 = -\nu(x_2^3 - \alpha x_2 - y_2), \quad \dot{y}_2 = x_2 - y_2 - \underline{z_1(t)}, \quad \dot{z}_2 = \beta y_2, \tag{5}$$

$$\dot{x}_2 = -\nu[x_2^3 - \alpha \underline{x_1(t)} - y_2], \quad \dot{y}_2 = x_2 - y_2 - z_2, \quad \dot{z}_2 = \beta y_2, \tag{6}$$

The transverse Lyapunov spectra corresponding to all these connections, together with the Lyapunov exponents for the original chaotic system (1), are reported in Table I.

III. RESULTS

A. Networks with homogeneous driving

The results presented in this work correspond to small arrays with four VPD circuits that are in the chaotic regime, starting with the case of four identical units connected in cascade, as described by the following set of equations,

$$\begin{aligned}&\dot{x}_1 = -\nu(x_1^3 - \alpha x_1 - y_1), \quad \dot{y}_1 = x_1 - y_1 - z_1, \quad \dot{z}_1 = \beta y_1, \quad \dot{x}_2 = -\nu(x_2^3 - \alpha x_2 - y_2), \quad \dot{y}_2 = \underline{x_1(t)} - y_2 - z_2, \quad \dot{z}_2 = \beta y_2,\\ &\dot{x}_3 = -\nu(x_3^3 - \alpha x_3 - y_3), \quad \dot{y}_3 = \underline{x_2(t)} - y_3 - z_3, \quad \dot{z}_3 = \beta y_3, \quad \dot{x}_4 = -\nu(x_4^3 - \alpha x_4 - y_4), \quad \dot{y}_4 = \underline{x_3(t)} - y_4 - z_4, \quad \dot{z}_4 = \beta y_4,\end{aligned} \tag{7}$$

the results being presented in graphical form in Fig. 3. One could think that the behavior so far obtained should be trivially related to that presented in Fig. 1, but it can be seen that the second and third systems synchronize in the first place, while, later, the first and fourth oscillators become synchronized after a short transient during which these systems exhibit some kind of anti-phase-synchronous behavior.

The following setting to be considered consists in the competition of connections (2) and (5) in a network with four VPD oscillators:

$$\begin{aligned}&\dot{x}_1 = -\nu(x_1^3 - \alpha x_1 - y_1), \quad \dot{y}_1 = x_1 - y_1 - z_1, \quad \dot{z}_1 = \beta y_1, \quad \dot{x}_2 = -\nu(x_2^3 - \alpha x_2 - y_2), \quad \dot{y}_2 = \underline{x_1(t)} - y_2 - \underline{z_4(t)}, \quad \dot{z}_2 = \beta y_2,\\ &\dot{x}_3 = -\nu(x_3^3 - \alpha x_3 - y_3), \quad \dot{y}_3 = \underline{x_1(t)} - y_3 - \underline{z_4(t)}, \quad \dot{z}_3 = \beta y_3, \quad \dot{x}_4 = -\nu(x_4^3 - \alpha x_4 - y_4), \quad \dot{y}_4 = x_4 - y_4 - z_4, \quad \dot{z}_4 = \beta y_4,\end{aligned} \tag{8}$$

while the result can be found in Fig. 4. The result implies that the system achieves an asymptotically stable behavior, which does not depend on the initial condition that is chosen.

In this kind of *competitive* connection in which one oscillator is simultaneously driven by several different signals the observed behavior is some kind of *compromise* between the inputs that cannot be predicted *a priori*, but it appears that each connection has a sort of strength, and, hence, some of them may have the ability to dominate the others. Thus, in this case it is clear that connection (2) is able to overcome the instability associated with connection (5). Notice that by using the original PC method it would not be possible to establish this kind of connection for a three-dimensional circuit, because by injecting two signals one would have a one-dimensional, and hence trivial, circuit.

In the next connection to be considered four VPD oscillators are connected in a loop with two different (alternating) connections, the system being described by the following equations:

$$\dot{x}_1 = -\nu[x_1^3 - \alpha x - \underline{y_4(t)}], \quad \dot{y}_1 = x_1 - y_1 - z_1, \quad \dot{z}_1 = \beta y_1, \quad \dot{x}_2 = -\nu(x_2^3 - \alpha x_2 - y_2), \quad \dot{y}_2 = \underline{x_1(t)} - y_2 - z_2, \quad \dot{z}_2 = \beta y_2,$$

$$\dot{x}_3 = -\nu[x_3^3 - \alpha x_3 - \underline{y_2(t)}], \quad \dot{y}_3 = x_3 - y_3 - z_3, \quad \dot{z}_3 = \beta y_3, \quad \dot{x}_4 = -\nu(x_4^3 - \alpha x_4 - y_4), \quad \dot{y}_4 = \underline{x_3(t)} - y_4 - z_4, \quad \dot{z}_4 = \beta y_4, \tag{9}$$

while a numerical simulation is presented in Fig. 5. The two connections used in this case are both stable (see Table I).

It can be seen that in a first moment the VPD oscillators synchronize in pairs; namely, the second and third oscillators do synchronize in this example, while the same can be said about the first and fourth systems. Then, later, the four oscillators become completely coherent. The signals that drive each system imply a relatively strong perturbation in its behavior, and, thus, the system needs some time before converging towards the stable synchronized behavior among the four oscillators. However, the transient behavior in which one has that two pairs of oscillators synchronize separately in the first place appears to be *typical*, in the sense that one always finds it when starting with random initial conditions for the four oscillators, although, of course, the symmetry of the problem implies that it is not possible to know which will be the precise oscillators that exhibit transient synchronization one with another. In some simulations the transient appears to have a very long lifetime, but, apparently, for any set of initial conditions that lie in the attraction basin of the strange attractor before coupling, one always reaches the coherent cluster of four oscillators.

In the last setting that has been considered, four VPD oscillators are linked also in a loop and with two alternating connections, the array being described by the following equations:

$$\dot{x}_1 = -\nu[x_1^3 - \alpha \underline{x_4(t)} - y_1], \quad \dot{y}_1 = x_1 - y_1 - z_1, \quad \dot{z}_1 = \beta y_1, \quad \dot{x}_2 = -\nu[x_2^3 - \alpha x_2 - \underline{y_1(t)}], \quad \dot{y}_2 = x_2 - y_2 - z_2, \quad \dot{z}_2 = \beta y_2,$$

$$\dot{x}_3 = -\nu[x_3^3 - \alpha \underline{x_2(t)} - y_3], \quad \dot{y}_3 = x_3 - y_3 - z_3, \quad \dot{z}_3 = \beta y_3, \quad \dot{x}_4 = -\nu[x_4^3 - \alpha x_4 - \underline{y_3(t)}], \quad \dot{y}_4 = x_4 - y_4 - z_4, \quad \dot{z}_4 = \beta y_4. \tag{10}$$

The final, perhaps surprising, result is that the system attains a self-organized coherent *periodic* state (see Fig. 6). Thus, in this case one has spontaneous emergence of ordered behavior in a network of chaotic oscillators that are locally coupled. However, here the mechanism of the process is not the same as in the case of globally coupled maps, as reported by Kaneko [13], and the explanation is as follows. Connection (6) is nonsynchronizing (it even has two unstable directions, as shown in Table I). However, the behavior along the unstable directions is not necessarily unbounded in this case, as it makes the system fall outside the attraction basin of the strange attractor and perform a transition to a stable limit cycle that coexists with the strange attractor. Thus, in this case the mechanism of the phenomenon is not purely dynamical as in the examples reported in Refs. [13,17,19].

B. Networks with inhomogeneous driving

In the second part of this work we have considered networks in which one of the oscillators has different values of parameters. In particular, these oscillators will be in a periodic window or on the road to chaos, but the initial conditions will not be such that the system goes to the competing limit cycle. This study can be interesting from the purely theoretical side, and, in addition, this can be considered a first step in analyzing the behavior of the network in the case where some external input acts on some oscillator changing its state.

The first connection that has been considered corresponds to a cascade of four elements formed with connection (2), that is, the same connection described by Eq. (7) in Sec. III A. The first oscillator has been chosen such that it is in a periodic orbit that spans both scrolls and that is in the periodic window described in Sec. II namely, it has $\beta = 500$). The result (see Fig. 7) is that this setting induces a transition from the strange attractor to the coexisting stable limit cycle in the three cascaded oscillators. The same behavior can be simply obtained by adding some form of external noise that will assist in the transition between the two attractors, indicating that, with respect to the chaotic signal produced by an identical system (see Fig. 3), the signal corresponding to another parameter value is equivalent to a *noisy* signal.

Now, we shall consider the case of the connection reported in Eq. (8), in which two oscillators are driven simultaneously by two signals that enter through different connections. Here we shall consider the case in which the two driven oscillators are still chaotic, while the driving signals correspond to systems that are different, although they share the feature of being in the periodic regime (see Fig. 8). The result is analogous to the one reported in Fig. 4 in the sense that the behavior of the driven systems is a compromise between their own dynamics and the periodic driving signals, although the behavior appears to be now periodic, albeit complex.

In turn, we shall now consider the already mentioned situation of the setting (9), in which two different alternate connections coexist. As was already mentioned in Sec. III A, the

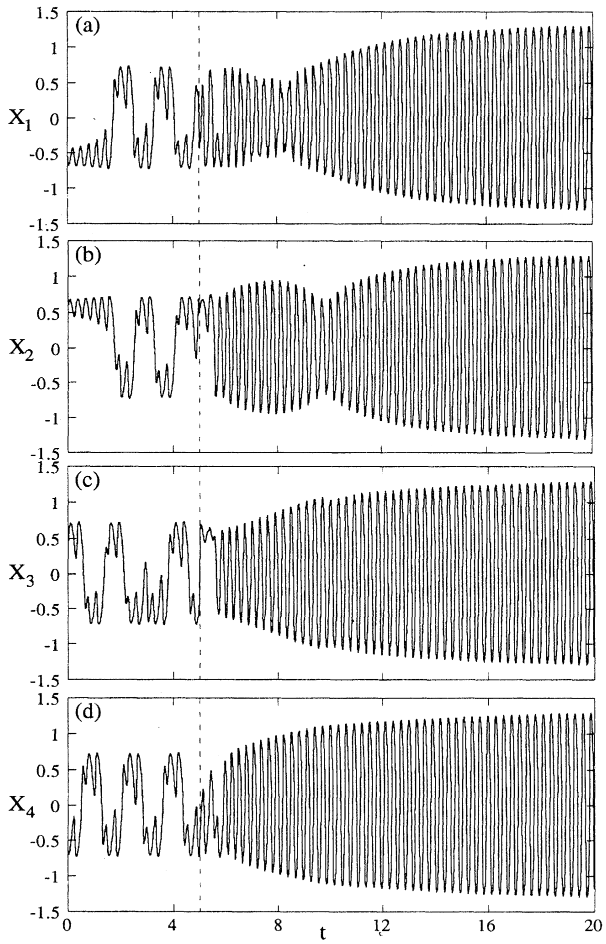

FIG. 6. Four van der Pol–Duffing oscillators (1) connected in a closed loop according to Eq. (10), such that connections (3) and (6) alternate. See Fig. 1 for the parameters.

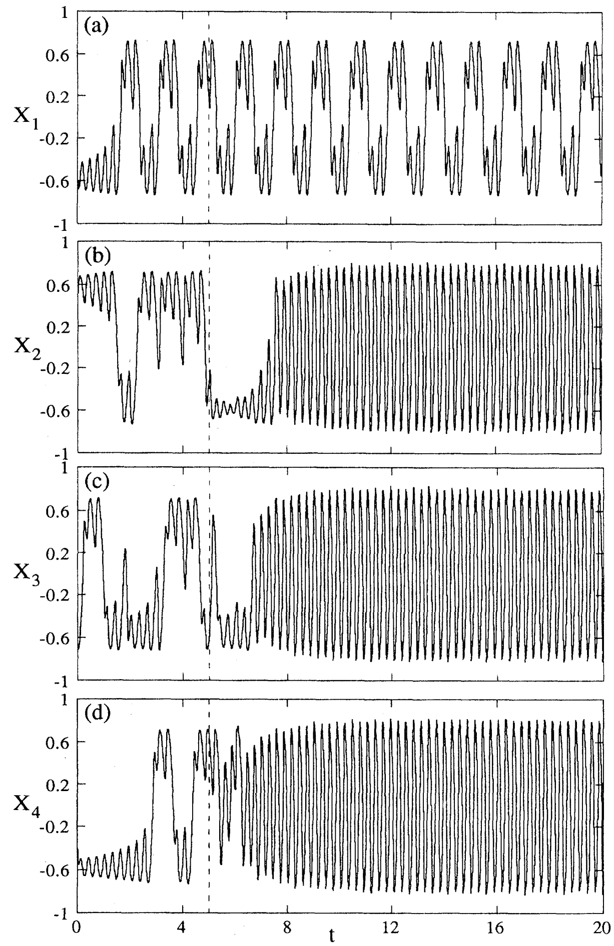

FIG. 7. Cascade of four van der Pol–Duffing systems (1) [see Eq. (7) for the setting], such that the first oscillator is periodic ($\nu=100$, $\alpha=0.35$, and $\beta=500$), while the other oscillators have the same parameter values given in Fig. 1. Compare with the results obtained in Fig. 3 in the case of homogeneous driving.

behavior of the network in the case where one of the systems is different differs from the one reported in Fig. 5 (the four oscillators did synchronize) in the fact that now symmetry has been broken, and the globally synchronized cluster splits into two clusters with two oscillators each (see Fig. 9). This behavior was observed as a transient in the case in which the four oscillators are identical, but now becomes the most stable behavior. However, the globally synchronized behavior appears as a transient behavior in many runs (as seen in Fig. 9), although ultimately the system sets up in the two-cluster behavior. Sometimes the long-time limit does not set up easily, as the transient with coherent chaotic behavior extends till very long times. It has been checked that this is indeed the globally attracting state, both by running some very long simulations and by adding some small amount of external white noise to the system, such that the intrinsic bimodality of this situation can be more easily resolved. In the case of identical oscillators the addition of this noise yields a globally synchronized state, thus introducing no bias towards the symmetry-broken solution.

Finally, we shall consider again the setting of (10), that is, the connection in which one has two alternating connections, one of them being unstable from the point of view of synchronization. In Fig. 6 it was shown that the presence of connection (6) makes the system perform a transition to a globally synchronized regular state, this state being formed by four synchronized limit cycles. In the behavior shown in Fig. 10 the first oscillator has $\beta=1000$, and one has that the four systems are in the periodic state, but now with different frequencies and/or phases in their oscillations, i.e., they are not synchronized. The system does not perform a sudden transition between the completely synchronized state of Fig. 6 and the behavior of Fig. 10. Instead, if one stays closer to the homogeneous case, namely by using $\beta=500$, one has four periodic oscillators that have the same frequency but are split into two clusters of two, the oscillators being synchronized within each cluster, causing the two clusters to differ by a phase. As one has limit cycles that have approximately the same amplitude, it is not surprising to find that the behavior resembles the one found for the so-called phase model [23] that is widely used in the modeling of neural networks, heart tissue, chemical reactions, etc.

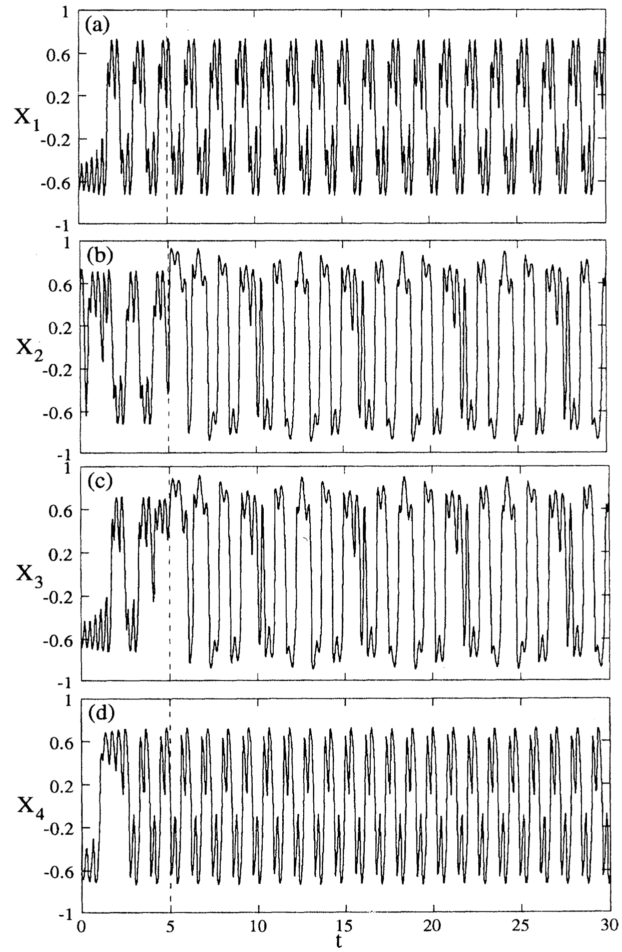

FIG. 8. Four van der Pol-Duffing systems (1) connected in competition according to Eq. (8), as the results reported in Fig. 4, but now the first oscillator has $\nu = 100$, $\alpha = 0.35$, and $\beta = 500$, while the last has $\nu = 100$, $\alpha = 0.35$, and $\beta = 305$. See Fig. 1 for the parameters of the other two oscillators.

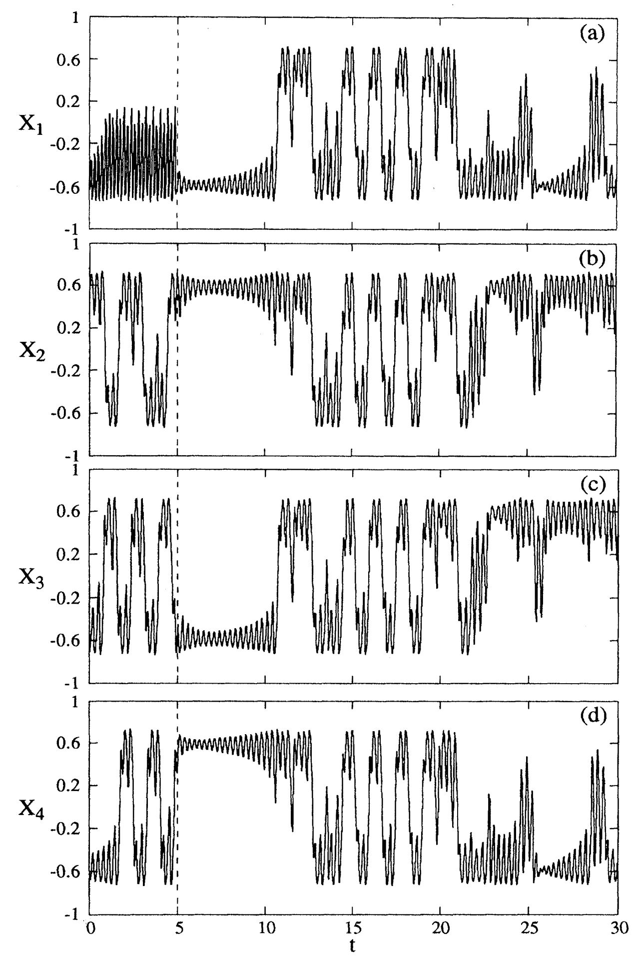

FIG. 9. Four van der Pol-Duffing oscillators (1) connected in a closed loop according to Eq. (9) (see Fig. 5 for more details), such that the first oscillator has $\nu = 100$, $\alpha = 0.35$, and $\beta = 1000$. See Fig. 1 for the parameters of the other three oscillators.

IV. CONCLUSIONS

The aim of the present contribution has been to exploit some of the inherent potentialities of a recently suggested method [11] that allows one to obtain synchronized behavior in chaotic systems through one-way connections. The method can be very useful in the case of low-dimensional systems because it allows one to devise networks composed of many chaotic units, so that many possible interconnections are allowed. Results are presented in the present work for the case of a Van der Pol–Duffing circuit [12] that is suitable for its implementation in an analog circuit.

A few different connections have been considered for the case of four oscillators, starting with a cascade, in which a set of oscillators is connected in a linear geometry (with no connection between the two ends) by using a single connection. Then, the situation in which two oscillators receive input simultaneously from two other systems is considered. A different connection consists of four oscillators connected in a closed loop with alternating stable connections (from the point of view of synchronization), the result also being synchronized behavior. The last connection considered is also a closed loop with two alternating connections, but one of them now being unstable from the point of view of synchronization. The behavior observed is a periodic collective state, the mechanism being the transition to a competing stable limit cycle induced by the unstable connection.

The consideration of inhomogeneous driving, i.e., the fact that some of the oscillators are different from the others, yields some qualitatively different behaviors, the transition to the coexisting limit cycle in the case of the linear cascade, where the dissimilarity in the signals can be considered as some external noise, or the symmetry breaking in two clusters in the case of the alternating closed loop with stable connections. In the case of the latter geometry, but with the consideration of one unstable connection, one has a transition to two clusters that differ by a phase, or a set of different periodic behaviors, depending on the parameter differences.

In addition to the interest that the kinds of systems studied in this work may generate on their own, coupled chaotic systems are already interesting from the viewpoint of dy

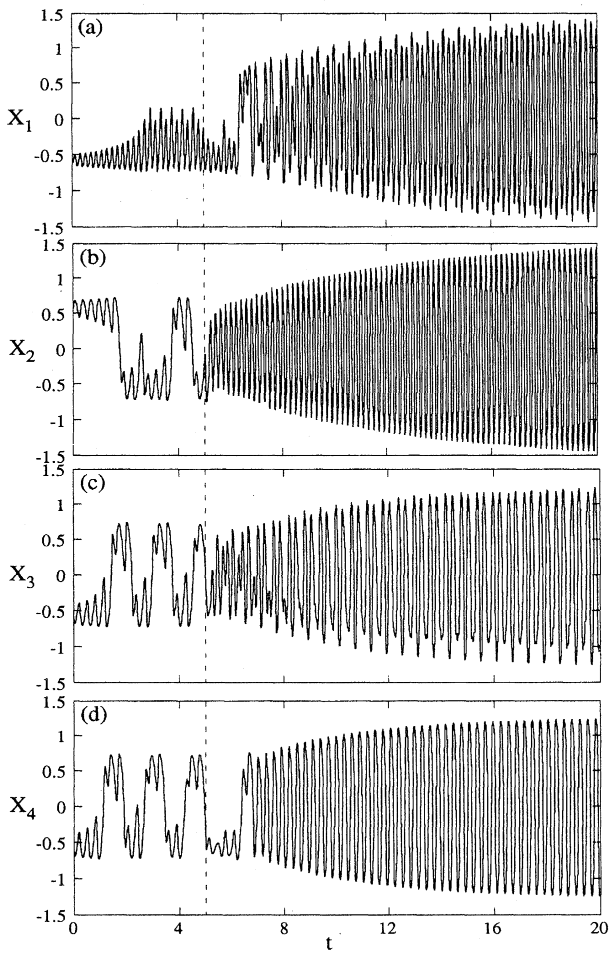

FIG. 10. Four van der Pol–Duffing oscillators (1) connected in a closed loop according to Eq. (10) (see Fig. 10 for more details), such that the first oscillator has $\nu=100$, $\alpha=0.35$, and $\beta=1000$. See Fig. 1 for the parameters of the other three oscillators.

namical systems theory. These systems may offer many new and rich dynamical behaviors, compared to those already found in dimensions three or four. For instance, by considering systems of dimensions five or six that possess some elements of symmetry, it has been shown [24] that new behaviors can be found, such as riddled basins [25], with an extreme dependence on the initial conditions, or on-off intermittency [26], with short excursions out of a stable state. Symmetry is an important ingredient in the case of these new behaviors, and, thus, it would not be surprising to find some new behaviors in the case of four three-dimensional systems; as has been shown in the present work, some connections might support both states with one set of four and two sets of two synchronized oscillators.

On the other hand, Kaneko [13] has recently introduced the idea that switching among different coexisting clusters can be a form of encoding information in a chaotic network. In his work he has considered simple quadratic maps connected through global coupling, while the choice of the adequate state of the system is achieved by changing the coupling among the maps. In the framework of the connection method considered in the present work, there are no parameters regulating this interaction, the connectivity being related, instead, to which units are connected and through which connection. This could be a more realistic representation of the way in which real neurons interact, especially if one uses continuous or discrete neuron models to represent the local units [27].

ACKNOWLEDGMENTS

This work was supported in part by DGICYT (Spain) Research Grants No. PB92-0279 (J.G.) and No. PB92-0295 (M.A.M.). One of us (J.G.) also acknowledges partial support from *Caja de Ahorros de Santander y Cantabria*.

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